DISCRETE MATHEMATICS

On vertex-degree restricted paths in polyhedral graphs

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Abstract

It is proved that every 3-connected planar graph G with $\delta(G)\geqslant 4$ either does not contain any path on $k\geqslant 8$ vertices or must contain a path on k vertices ($k\geqslant 8$) having degree (in G) at most $5k-7$; the bound $5k-7$ is shown to be the best possible. For every connected planar graph H different from a path and for every integer $m\geqslant 4$ there is a 3-connected planar graph G with $\delta(G)\geqslant 4$ such that each subgraph of G isomorphic to H has a vertex x with $\deg_G(x)\geqslant m$.

Keywords: 3-connected planar graph; 4-connected planar graph; Subgraph with restricted degrees; Path

1. Introduction

Throughout this paper we shall consider planar graphs without loops and multiple edges. We use the standard terminology and notation, see, e.g. [1]. Let us recall, however, more specialized notations. By a plane graph or, equivalently, by a plane map we mean an embedding of a planar graph in the plane. The *degree* of a face α of a plane graph is the number of edges incident with α where each cut-edge is counted twice. Vertices and faces of degree i are called *i-vertices* and *i-faces*, respectively. For a plane graph G let $V(G)$, $E(G)$ and $F(G)$ denote the vertex-set, the edge-set and the face-set of G, respectively. The degree of a vertex x (a face α) in G is denoted by $\deg_G(x)$ ($\deg_G(\alpha)$). $\delta(G) := \min\{\deg_G(x)\colon x \in V(G)\}$ is called the *minimum degree* of G.

For $i\geqslant 0$, let $v_i(G)$ denote the number of i-vertices of G. Similarly, $f_i(G)$, $i\geqslant 3$, will stand for the number of i-faces of G. If G is a plane graph with t components

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then

$$\sum_{i\geqslant 0}(6-i)v_i(G)+2\sum_{i\geqslant 3}(3-i)f_i(G)=6(1+t) \tag{1}$$

can be derived from a well-known formula of Euler. A path on k vertices is called a *k-path.* A k-path is denoted by $P=P_k=[x_1,x_2,\ldots,x_k]$ and is said to be a (x_1,x_k)*-path.* Two (x,y)-paths P and Q are *i-disjoint* if they have no inner vertices in common, i.e. if $V(P)\cap V(Q)=\{x,y\}$. Let $\mathrm{Med}(G)$ denote the graph resulting from a 3-connected plane graph G by subdividing each edge of G by a new vertex and joining two new vertices x, x' if the corresponding edges e, $e'\in E(G)$ are adjacent boundary edges of a common face of G. Let in the sequel $\mathrm{Med}^{(n)}(G):=\mathrm{Med}(\mathrm{Med}^{(n-1)}(G))$, $n=2,3,\ldots,$ where $\mathrm{Med}^{(1)}(G)=\mathrm{Med}(G)$.

It is a classical corollary of Euler's famous formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [10,11] states that every 3-connected planar graph G contains an edge with degree-sum of its endvertices no larger than 13 and, more special, no larger than 11 if $\delta(G)\geqslant 4$ (see also [3,4,6,9,13]). These bounds are best possible. This result was further strengthened in various directions and has served as a starting point for discovering many structural properties of embeddings of graphs, see e.g. [5,6,13]. In [7,8] it has been proved results similar to the one due to Kotzig for k-paths ($k=3,4,5$) in 3-connected plane graphs. On the other hand, in [7] it has been proved that for every pair of integers k and m ($k,m\geqslant 3$) there exists a 2-connected planar graph G containing a k-path in which every k-path contains a vertex x such that $\deg_G(x)\geqslant m$.

These results suggest the following problem. Let positive integers c,δ be given where $3\leqslant c\leqslant \delta\leqslant 5$. For a connected planar graph H, let $\mathscr{G}(c,\delta;H)$ denote the family of all c-connected planar graphs with $\delta(G)\geqslant\delta$ having a subgraph isomorphic to H.

Problem. *What is the minimum integer* $\varphi(c,\delta;H)$ *such that every graph* $G\in\mathscr{G}(c,\delta;H)$ *contains a subgraph* H' *isomorphic to* H *for which*

$$\deg_G(x)\leqslant\varphi(c,\delta;H)$$

holds for every $x\in V(H')$?

In [2] has been proved

(i) $\varphi(3,3;P_k)=5k,\ k\geqslant 1$,

(ii) $\varphi(3,3;H)=\infty$ for any $H\neq P_k$.

For convenience put $\mathscr{G}(c,\delta;k):=\mathscr{G}(c,\delta;P_k)$ and $\varphi(c,\delta;k):=\varphi(c,\delta;P_k)$.

For all c, δ with $3\leqslant c\leqslant\delta\leqslant 5$ we have $\varphi(c,\delta;1)=5$. Here the relation $\delta(G)\leqslant 5$ for every planar graph G implies $\varphi(c,\delta;1)\leqslant 5$, and considering the icosahedron graph we obtain $\varphi(c,\delta;1)\geqslant 5$.

The aim of this paper is to determine $\varphi(3,4;k)$ and to give lower and upper bounds for $\varphi(4,4;k)$. Moreover, $\varphi(3,4;H)=\infty$ is proved for any connected planar graph $H\neq P_k$ ($k\geqslant 1$), which means that for any given integer $m\geqslant 4$, there exists a graph

$G \in \mathcal{G}(3,4;H)$ such that every subgraph H' of G isomorphic to H contains a vertex $x \in V(H')$ satisfying $\deg_G(x) \geqslant m$.

2. Results

Theorem 1. (i) $\varphi(3,4;1) = 5$, (ii) $\varphi(3,4;2) = 7$, (iii) $\varphi(3,4;3) = 9$, (iv)$15 \leqslant \varphi(3,4;4) \leqslant 20$, (v) $19 \leqslant \varphi(3,4;5) \leqslant 25$, (vi) $\varphi(3,4;6) = 23$, (vii) $27 \leqslant \varphi(3,4;7) \leqslant 28$, (viii) $\varphi(3,4;k) = 5k - 7$ *for* $k \geqslant 8$.

Remark 1. In fact, one can even show $\varphi(3,4;4)=15, \varphi(3,4;5)=19$ and $\varphi(3,4;7)=27$, but in view of the size of this paper we omit the proofs.

Theorem 2. (i) $6 \leqslant \varphi(4,4;2) \leqslant 7$, (ii) $9 \leqslant \varphi(4,4;3) \leqslant 10$, (iii) $\max\{5\lfloor(3k+1)/11\rfloor + 5, 3k - 6\lceil(3k+1)/11\rceil + 6\} \leqslant \varphi(4,4;k) \leqslant 3k+1$ *for* $k \geqslant 4$.

Remark 2. Note that

$$\max\left\{5\left\lfloor \frac{3k+1}{11}\right\rfloor + 5,\ 3k - 6\left\lceil\frac{3k+1}{11}\right\rceil + 6\right\}$$

$$= \begin{cases} 5\lfloor\frac{3k+1}{11}\rfloor + 5 & \text{if } 0 \leqslant r \leqslant 6, \\ 3k - 6\lceil\frac{3k+1}{11}\rceil + 6 & \text{if } 6 < r < 11 \end{cases}$$

is true for $k \geqslant 4$ with $3k+1 \equiv r(11)$.

Theorem 3. $\varphi(3,4;H) = \infty$ *for every connected planar graph* $H \neq P_k$ $(k \geqslant 1)$.

3. Proofs

Proof of Theorem 1. Parts (i) and (ii) of Theorem 1 are easy consequences of Euler's formula and Kotzig's work, respectively. Part (iii) is treated in [7] and the upper bounds of parts (iv) and (v) can be found in [2].

I. To prove the lower bounds it is enough to exhibit a 3-connected plane graph G with minimum degree $\delta(G) \geqslant 4$ in which every k-path contains a vertex of degree at least $l(k)$, where $l(4)=15$, $l(5)=19$, $l(6)=23$, $l(7)=27$ and $l(k)=5k-7$ for $k \geqslant 8$.

The construction starts with the graph of the dodecahedron. Into each of its 5-faces we insert a new vertex and join it to the 5 original vertices incident with this face, so we obtain a triangulation T. Let

$$H := \begin{cases} T & \text{if } k = 4, \\ \mathrm{Med}(T) & \text{if } k = 5 \text{ or } k \geqslant 8, \\ \mathrm{Med}^{(2)}(T) & \text{if } k \in \{6,7\}. \end{cases}$$

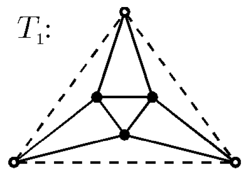

Fig. 1.

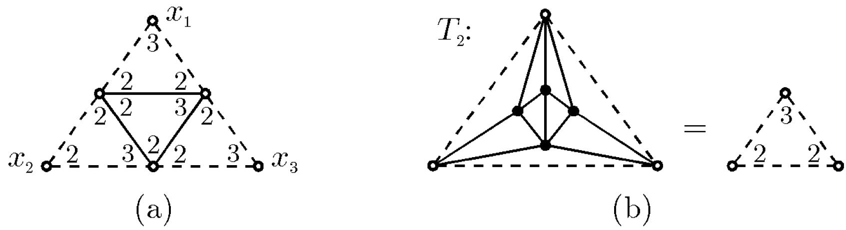

Fig. 2.

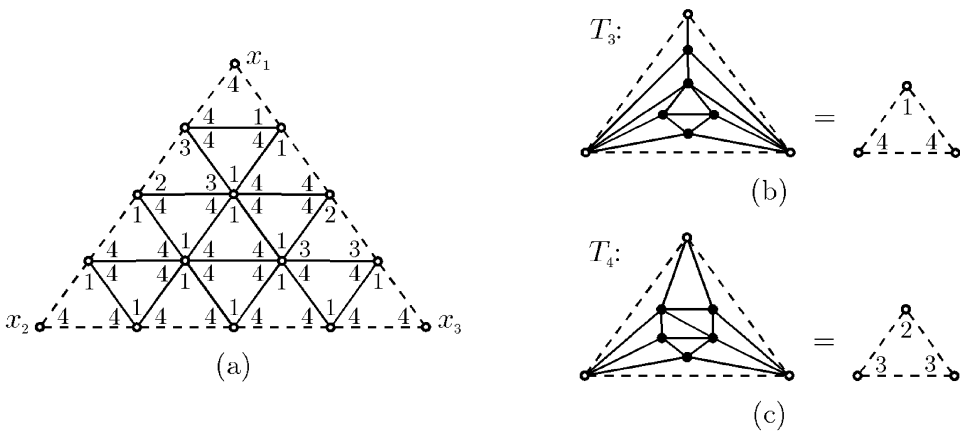

Fig. 3.

Replace each 3-face of H by the following configuration, according to the case we are in:

(iv) $(k = 4)$ by T_1 (Fig. 1)

(v) $(k = 5)$ by T_2 (Fig. 2b)

(vi) $(k = 6)$ by T_3 (Fig. 3b) or by T_4 (Fig. 3c), resp.

(vii) $(k = 7)$ by T_5 (Fig. 4b) or by T_6 (Fig. 4c), resp.

(viii) $(k \geqslant 8)$ by T_7 (Fig. 5c, T_7 consists of $k - 1$ (black) vertices) or by T_8 (Fig. 5d, T_8 consists of $k - 1 + \lfloor (k-2)/2 \rfloor$ (black) vertices), respectively.

The Figs. 2a $(k=5)$, 3a $(k=6)$, 4a $(k=7)$, 5a $(k \geqslant 8$, k even) and 5b $(k \geqslant 8$, k odd) respectively show the orientation of T_i $(i \in \{2, 3, \ldots, 8\})$ in H, where $x_1x_2x_3 \in F(T)$ such that $\deg_T(x_1) = 5$, $\deg_T(x_2) = \deg_T(x_3) = 6$ and $x_1x_2x_3$ is positively oriented.

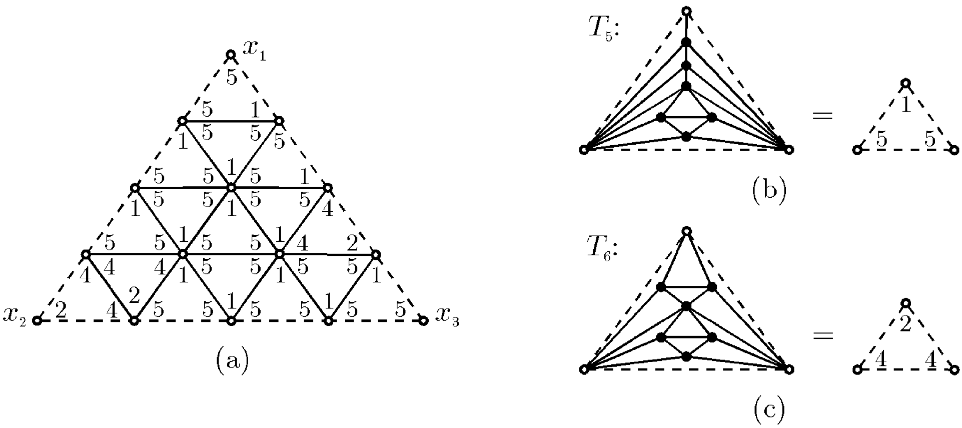

Fig. 4.

II. It remains to show that $\varphi(3,4;k) \leqslant 5k-7$, $k \geqslant 6$.

Supposing our theorem is not true, that means there is a $k \geqslant 6$ such that $\varphi(3,4;k) \geqslant 5k-6$. Let G be a counterexample with n vertices and a maximum number of edges. A vertex $x \in V(G)$ is a *major vertex* (*minor vertex*) if $\deg_G(x) \geqslant 5k-6$ ($\deg_G(x) < 5k-6$).

Property 1. *Each k-path of G contains a major vertex*

Property 2. *Any r-face* $\alpha \in F(G)$, $r \geqslant 4$, *is incident with minor vertices only* (by maximality of the number of edges, otherwise one could add a diagonal).

Let $M = M(G)$ be the plane graph induced by the set of major vertices of G. For $\alpha \in F(M)$ let G_α be the component induced by all vertices of $V(G) \setminus V(M)$ embedded in the interior of α.

Property 3. *All vertices of* G_α *are minor vertices.*

Property 4. *There is no k-path in* G_α.

Property 5. *Let* $x \in V(M)$, $\alpha \in F(M)$ *incident with x and let* $\{u_1, u_2, \ldots, u_m\} \subseteq V(G) \setminus V(M)$ *be the set of neighbours of x in* α. *There is a path* $[u_1, u_2, \ldots, u_m]$ *in* α (because of Property 2), *see Fig.* 6.

Property 6. $v_i(M) = 0$, $i \leqslant 3$ (as otherwise $\deg_G(x) \leqslant i + i(k-1) \leqslant 3 + 3(k-1) < 5k-6$ for some major vertex $x \in V(M)$).

Let $\alpha = x_1x_2x_3 \in F(M)$ and $u_1, u_2, \ldots, u_m$ incident with x_1 in G_α. Property 2 implies u_1x_3, $u_mx_2 \in E(G)$ (see Fig. 7). We denote by $P_{i,j}$ $(1 \leqslant i < j \leqslant m)$ the path $[u_i, u_{i+1}, \ldots, u_{j-1}, u_j]$.

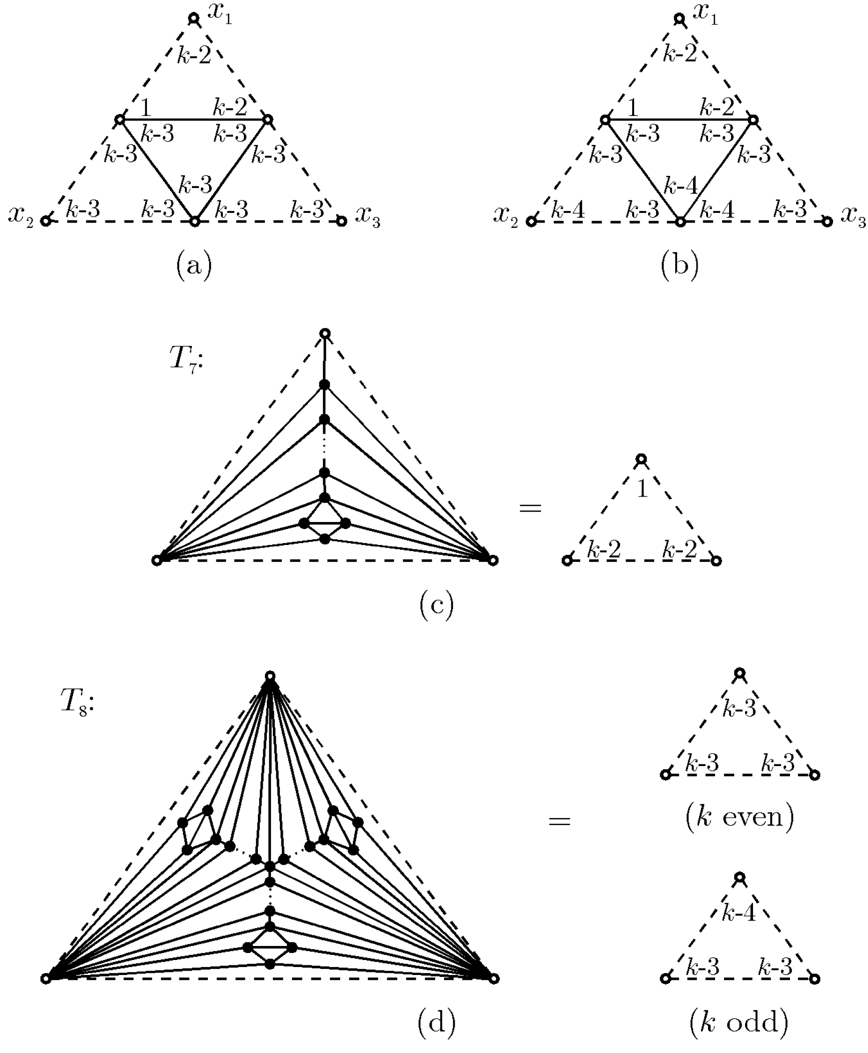

Fig. 5.

Lemma 1. *If there is an edge $u_iu_j \in E(G_\alpha)$ with $j-i \geqslant 2$, then there is a (u_i,u_j)-path P_μ in G_α with $\mu \geqslant j-i+2$ containing only vertices on the border and in the interior of the cycle $C = P_{i,j} \cup \{u_iu_j\}$.*

Proof. Let r,s $(i \leqslant r < s \leqslant j)$ be integers such that there is a (u_r,u_s)-path $Q_{r,s} = [u_r = w_1, w_2, \ldots, w_l = u_s]$ which is i-disjoint to $P_{r,s}$ and all vertices of $Q_{r,s}$ are on the border or in the interior of C (where perhaps $Q_{r,s} = [u_r,u_s]$) and $s-r$ is minimal.

Case 1: $s-r=1$. Since there are no multiple edges, $Q_{r,s}$ has more edges than the one edge of $P_{r,s}$ and $P_\mu := P_{i,r} \cup Q_{r,s} \cup P_{s,j}$ satisfies the requirements of our lemma.

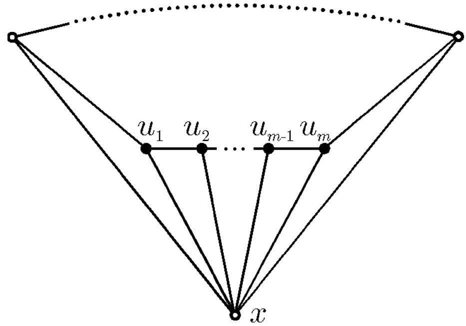

Fig. 6.

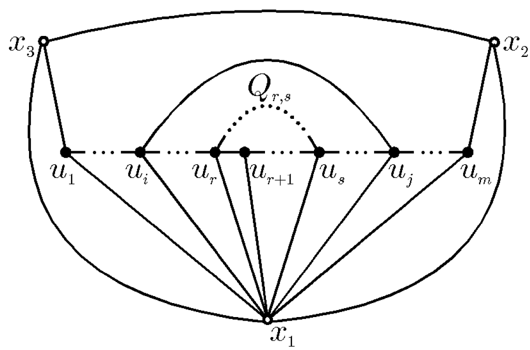

Fig. 7.

Case 2: $s - r \geqslant 2$ (see Fig. 7). Each vertex has at least four neighbours ($\delta(G) \geqslant 4$). That means u_{r+1} has a neighbour $z \notin \{u_r, u_{r+2}, x_1\}$ and z lies on the border or in the interior of the cycle $C' = P_{r,s} \cup Q_{r,s}$. Because of the 3-connectivity of a polyhedral graph we can find a path P from z to a vertex z' of C', $z' \neq u_{r+1}$ (where potentially $z = z' \in V(Q_{r,s})$), such that no inner vertex of P belongs to C'. z' is an inner vertex w_q of $Q_{r,s}$ (as otherwise $s - r$ is not minimal). The path $P' := [u_{r+1}, z] \cup P \cup [z' = w_q, w_{q+1}, \ldots, w_{l-1}, w_l = u_s]$ of G_α has no inner vertex in common with $\{u_1, u_2, \ldots, u_m\}$ and contradicts the minimality of $s - r$. $\square$

For $x \in V(M)$, $\alpha \in F(M)$, $x \in \alpha$, we define

$$w(x, \alpha) := |\{u \in V(G_\alpha) \colon ux \in E(G)\}|,$$

$$b(x, \alpha) := \begin{cases} w(x, \alpha) - (k - 3) & \text{for } w(x, \alpha) > k - 3, \\ 0 & \text{for } w(x, \alpha) \leqslant k - 3, \end{cases}$$

$$b(\alpha) := \sum_{x\colon\, x \in \alpha} b(x, \alpha), \qquad b(x) := \sum_{\alpha\colon\, x \in \alpha} b(x, \alpha).$$

Lemma 2. *For any triangle* $\alpha = x_1x_2x_3 \in F(M)$ *we have* $b(x_i,\alpha) \leqslant 1$ *for all* $i \in \{1,2,3\}$.

Proof. Suppose there is a triangle $\alpha = x_1x_2x_3 \in F(M)$ and $b(x_1,\alpha) = w(x_1,\alpha) - (k-3) \geqslant 2$. Then $w(x_1,\alpha) \geqslant k-1$. Let $\{u_1,u_2,\ldots,u_m\}$, $m = w(x_1,\alpha) \geqslant k-1$, be the set of neighbours of x_1 in α. Because of Properties 4 and 5 we have $m \leqslant k-1$. Hence $m = k-1$ and $b(x_1,\alpha) = 2$. If there is a $u \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-1}\}$ and $uu_1 \in E(G_\alpha)$ or $uu_{k-1} \in E(G_\alpha)$ then there is a k-path in G_α consisting of minor vertices only (by Property 3). Because of planarity the edges u_1x_2 and $u_{k-1}x_3$ cannot exist simultaneously in G. We may assume that u_1 has a neighbour $u' \in \{u_3,u_4,\ldots,u_{k-1}\}$ ($\delta(G) \geqslant 4$). Using Lemma 1 we find a k-path in G_α consisting of only minor vertices. This is a contradiction. □

Lemma 3. *Let* $\alpha = x_1x_2x_3 \in F(M)$ *be a triangle with* $b(x_1,\alpha) = 1$. *Then either* $w(x_2,\alpha) = 1$ *or* $w(x_3,\alpha) = 1$ *or* $w(x_2,\alpha) = w(x_3,\alpha) = 2$.

Proof. $b(x_1,\alpha) = 1$ means $w(x_1,\alpha) = k-2$.

Case 1: $u_1x_2 \in E(G)$ (or symmetrically $u_{k-2}x_3 \in E(G)$).

1.1. $u_1x_2x_3 \in F(G)$: Then we have $w(x_3,\alpha) = 1$.

1.2. $u_1x_2x_3 \notin F(G)$: Then, there is $u \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-2}\}$ with $u_1u \in E(G_\alpha)$, $ux_3 \in E(G)$ and $u_1ux_3 \in F(G)$. This implies $uu_{k-2} \notin E(G)$ because the cycle $C = [u_1,x_2,x_3]$ separates u and u_{k-2}. We already found a $(k-1)$-path $[u,u_1,u_2,\ldots,u_{k-2}]$ in G_α. There is no $z \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-2}\}$ with $zu_{k-2} \in E(G)$: otherwise there would be a k-path in G_α. As $\delta(G) \geqslant 4$ there is an edge $u_{k-2}u_i$ $(1 \leqslant i \leqslant k-4)$. Using Lemma 1 we find a (u_i,u_{k-2})-path $Q_{i,k-2}$ in G_α with at least $(k-2)-i+2$ vertices containing no vertex of $\{u_1,u_2,\ldots,u_{i-1}\}$. Together with the path $[u,u_1,u_2,\ldots,u_i]$ we have a k-path in G_α, a contradiction.

Case 2: $u_1x_2 \notin E(G)$ and $u_{k-2}x_3 \notin E(G)$.

2.1. There is a $u \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-2}\}$ with $uu_1 \in E(G_\alpha)$ and $uu_{k-2} \in E(G_\alpha)$: Then there is no $z \in V(G_\alpha) \setminus \{u,u_1,u_2,\ldots,u_{k-2}\}$ with $zu_1 \in E(G)$ or $zu_{k-2} \in E(G)$ or $zu \in E(G)$: otherwise there would have to be a k-path in G_α. So we have ux_2x_3, u_1ux_3, $u_{k-2}ux_2 \in F(G)$ and $w(x_2,\alpha) = w(x_3,\alpha) = 2$.

2.2. There is a $u \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-2}\}$ with $uu_1 \in E(G_\alpha)$ and $uu_{k-2} \notin E(G_\alpha)$ (symmetrically the case $uu_{k-2} \in E(G)$ and $uu_1 \notin E(G)$): Then there is no $z \in V(G_\alpha) \setminus \{u,u_1,u_2,\ldots,u_{k-2}\}$ with $zu_{k-2} \in E(G)$, otherwise there is a k-path in G_α. Because of $\delta(G) \geqslant 4$ there is an $i\,(1 \leqslant i \leqslant k-4)$ with $u_iu_{k-2} \in E(G)$. According to Lemma 1 there is a (u_i,u_{k-2})-path $Q_{i,k-2}$ in G_α with at least $k-2-i+2$ vertices containing no vertex of $\{u_1,u_2,\ldots,u_{i-1}\}$. $[u,u_1] \cup P_{1,i} \cup Q_{i,k-2}$ is a path in G_α with at least k vertices, a contradiction.

2.3. For each $u \in V(G_\alpha) \setminus \{u_1,u_2,\ldots,u_{k-2}\}$ we have $uu_1 \notin E(G)$ and $uu_{k-2} \notin E(G)$: This implies $u_1u_{k-2} \notin E(G)$, as otherwise $x_3u_1u_{k-2}x_2 \in F(G)$ is a 4-face in contradiction with Property 2. Therefore, there are integers i,j $(3 \leqslant i \leqslant j \leqslant k-4)$ with $u_1u_i \in E(G)$ and $u_ju_{k-2} \in E(G)$. By Lemma 1 there is a (u_1,u_i)-path $Q_{1,i}$ with at least $i-1+2$ vertices and a (u_j,u_{k-2})-path $Q_{j,k-2}$ with at least $k-2-j+2$ vertices containing no vertex of $\{u_{i+1},u_{i+2},\ldots,u_{k-2}\}$ and of $\{u_1,u_2,\ldots,u_{j-1}\}$, respectively.

$Q_{1,i} \cup P_{i,j} \cup Q_{j,k-2}$ is a path in G_α with at least $(i-1+2)+(j-i+1)+(k-2-j+2)-2=k$ vertices, a contradiction. $\square$

Lemma 4. *For any i-face* $\alpha \in F(M)$

$$b(\alpha) = \sum_{x:\, x\in\alpha} b(x,\alpha) \leqslant \begin{cases} 2i, & i \geqslant 4, \\ 2, & i = 3. \end{cases}$$

Proof. From the definition we have $b(x,\alpha) \leqslant (k-1)-(k-3) = 2$, which means $b(\alpha) \leqslant 2i$ for any i-face α. In case of a 3-face α we have $b(\alpha) \leqslant 2$ by Lemma 3. $\square$

We define

$$\mathbb{C} := \{\alpha = x_1x_2x_3 \in F(M) \colon b(\alpha) > 0 \text{ and } (\exists i \in \{1,2,3\}\colon w(x_i,\alpha) = 1)\},$$

$$\mathbb{D} := \{\alpha = x_1x_2x_3 \in F(M)\colon\ b(\alpha) > 0 \text{ and } (\exists i,j \in \{1,2,3\}, i \neq j\colon w(x_i,\alpha) = w(x_j,\alpha) = 2)\}$$

and for any $x \in V(M)$

$$c(x) := |\{\alpha \in \mathbb{C}\colon\ x \in \alpha,\ w(x,\alpha) = 1\}|,$$

$$d(x) := |\{\alpha \in \mathbb{D}\colon\ x \in \alpha,\ w(x,\alpha) = 2\}|.$$

$$c_i := \sum_{x:\ \deg_M(x)=i} c(x), \qquad d_i := \sum_{x:\ \deg_M(x)=i} d(x).$$

With $c := |\mathbb{C}|$ and $d := |\mathbb{D}|$ we have

$$c = \sum_{i\geqslant 3} c_i, \qquad 2d = \sum_{i \geqslant 3} d_i$$

and because of Property 6

$$c = \sum_{i\geqslant 4} c_i, \qquad 2d = \sum_{i \geqslant 4} d_i.$$

Lemma 5. *For any i-vertex* $x \in V(M)$, $i \geqslant 4$, *we have* $b(x) \geqslant 2c(x) + d(x) + 4(6-i)$.

Proof. In the case $i-(c(x)+d(x)) \geqslant 6$ we estimate: $b(x) \geqslant 0 \geqslant 4(6-i+c(x)+d(x)) \geqslant 4(6-i)+2c(x)+d(x)$.

We can assume $i-(c(x)+d(x)) \leqslant 5$. There are $c(x)$ faces α incident with x and $w(x,\alpha) = 1$; there are $d(x)$ faces α incident with x and $w(x,\alpha) = 2$; for the remaining $i-(c(x)+d(x))$ faces α incident with x we have $w(x,\alpha) \leqslant (k-3)+b(x,\alpha)$. Furthermore, there are i edges incident with x in M: $5k-6 \leqslant \deg_G(x) \leqslant c(x)+2d(x)+(i-(c(x)+d(x)))(k-3)+b(x)+i$. An easy estimation leads to $b(x) \geqslant 2c(x)+d(x)+4(6-i)$ using $k \geqslant 6$. $\square$

Lemma 6. *With*

$$S_f := 2c + d + \sum_{i \geqslant 4} 2if_i(M)$$

and

$$S_v := \sum_{i \geqslant 4} \left(\sum_{x:\ \deg_M(x)=i} (2c(x) + d(x) + 4(6-i)) \right)$$

we have $S_f \geqslant S_v$.

Proof. 1. We prove $S_f \geqslant \sum_{\alpha \in F(M)} b(\alpha)$. Any triangle $\alpha \in F(M)$ with a vertex x incident with α and $w(x,\alpha) = 1$ yields $b(\alpha) \leqslant 2$, the number of such triangles is c. Any triangle $\alpha \in F(M)$ with two vertices x, y incident with α and $w(x,\alpha) = w(y,\alpha) = 2$ yields $b(\alpha) \leqslant 1$, the number of such triangles is d. Any i-face $\alpha \in F(M)$ with $\deg_M(\alpha) = i \geqslant 4$ yields $b(\alpha) \leqslant 2i$ (Lemma 4). So we have

$$S_f = 2c + d + \sum_{i \geqslant 4} 2if_i(M) \geqslant \sum_{\alpha \in \mathbb{C}} b(\alpha) + \sum_{\alpha \in \mathbb{D}} b(\alpha) + \sum_{\alpha:\ \deg_M(\alpha) \geqslant 4} b(\alpha) = \sum_{\alpha \in F(M)} b(\alpha).$$

2. $\sum_{x \in V(M)} b(x) \geqslant S_v$ has been proved in Lemma 5.
3. Obviously, $\sum_{\alpha \in F(M)} b(\alpha) = \sum_{x \in V(M)} b(x)$. $\square$

Now we shall finish the proof of Theorem 1, part II.

$$\begin{aligned}
S_v - S_f &= \sum_{i \geqslant 4} \left(\sum_{x:\ \deg_M(x)=i} (2c(x) + d(x) + 4(6-i)) \right) - \left(2c + d + \sum_{i \geqslant 4} 2if_i(M) \right) \\
&= \sum_{i \geqslant 4} (2c_i + d_i + 4(6-i)v_i(M)) - \left(2c + d + 2\sum_{i \geqslant 4} if_i(M) \right) \\
&= 2c + 2d + 4\sum_{i \geqslant 4} (6-i)v_i(M) - 2c - d - 2\sum_{i \geqslant 4} if_i(M) \\
&= d + 4\sum_{i \geqslant 4} (6-i)v_i(M) - 2\sum_{i \geqslant 4} if_i(M) \\
&\geqslant 4\sum_{i \geqslant 4} (6-i)v_i(M) - 2\sum_{i \geqslant 4} if_i(M) - 6\sum_{i \geqslant 4} (i-4)f_i(M) \\
&= 4\sum_{i \geqslant 4} (6-i)v_i(M) + 8\sum_{i \geqslant 4} (3-i)f_i(M) \\
&\overset{(1)}{=} 24(1+t) \quad \text{(see the introduction)} \\
&> 0,
\end{aligned}$$

contradicting Lemma 6. This completes the proof of $\varphi(3,4;k) \leqslant 5k - 7$, for $k \geqslant 6$. $\square$

Proof of Theorem 2. First, let us prove the lower bound.

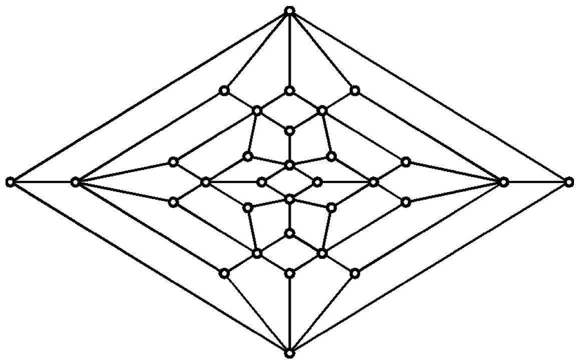

Fig. 8.

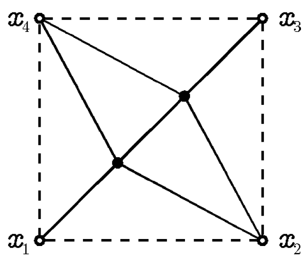

Fig. 9.

To see (i), place into every 4-face α of the cube graph a new vertex and join it with each vertex $x \in \alpha$. The so constructed graph G is 4-connected with $\delta(G)=4$ and each 2-path in G contains a vertex of degree 6.

To prove (ii), let H denote the rhombic triacontahedron as shown in Fig 8.

Every face $\alpha \in F(H)$ is a 4-face $\alpha = x_1x_2x_3x_4$ with

$$\deg_H(x_1) = \deg_H(x_3) = 5, \qquad \deg_H(x_2) = \deg_H(x_4) = 3.$$

Place two new adjacent vertices into every $\alpha = x_1x_2x_3x_4 \in F(H)$ and join them with the $x_i \in \alpha$ as presented in Fig. 9. Notice, that the so constructed graph G is 4-connected with $\delta(G)=4$. With $9 \leqslant \deg_G(x) \leqslant 10$, for each $x \in V(H)$, the construction rule implies that every 3-path of G contains a vertex of degree 9 or 10.

To prove (iii), insert into every face $\alpha = x_1x_2x_3x_4$ of the rhombic triacontahedron H of Fig. 8 a configuration T consisting of $k-1$ new vertices as is shown in Fig. 10.

Assume that x_1 and x_3 have the same number $i \geqslant 1$ of adjacent vertices in T where none of them are common neighbours. Moreover, let x_2 and x_4 have the same number $(k-1)-2i+2$ of adjacent vertices in T where $(k-1)-2i$ are common neighbours, i.e. $1 \leqslant i \leqslant (k-2)/2$. The so constructed graph G is 4-connected with $\delta(G)=4$. The

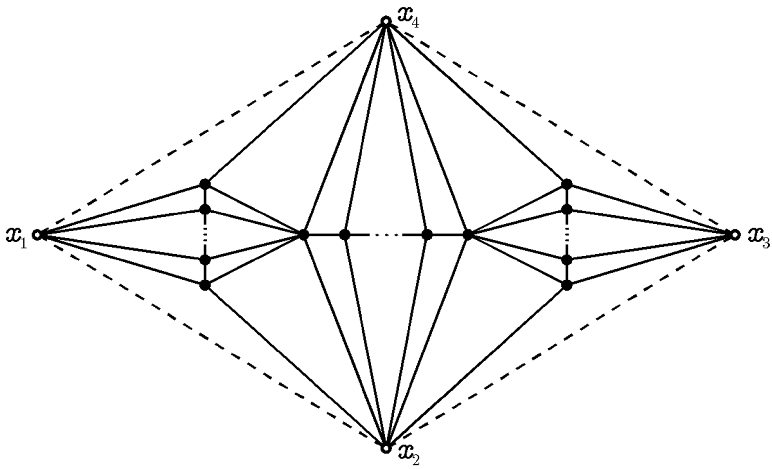

Fig. 10.

construction rule yields

$$\deg_G(x) = \begin{cases} 5i+5 & \text{if } \deg_H(x)=5, \\ 3k-6i+6 & \text{if } \deg_H(x)=3 \end{cases}$$

for each $x \in V(H)$. Thus, every k-path of G contains a vertex of degree no less than

$$\min\{5i+5, 3k-6i+6\}. \tag{2}$$

We maximize (2) by choosing a suitable i $(1 \leqslant i \leqslant (k-2)/2)$. Observe that $i = \lfloor (3k+1)/11 \rfloor$ is the largest integer such that $5i+5 \le 3k-6i+6$. Then,

$$\left\lfloor \frac{3k+1}{11} \right\rfloor \leqslant \frac{k-2}{2} \leqslant \left\lceil \frac{3k+1}{11} \right\rceil \quad \text{for } k=4,5$$

and

$$\left\lceil \frac{3k+1}{11} \right\rceil \leqslant \frac{k-2}{2} \quad \text{for } k \geqslant 6$$

can easily be confirmed which prove together with

$$5\left\lfloor \frac{3k+1}{11} \right\rfloor + 5 \geqslant 3k - 6\left\lceil \frac{3k+1}{11} \right\rceil + 6 \quad \text{for } k=4,5$$

the lower bound in (iii).

To prove the upper bound, suppose there is a k $(k \geqslant 2)$ such that $\varphi(4,4;k) \geqslant 3k+2$. Let, for such a k, G be a counterexample with n vertices and a maximum number of edges. A vertex $x \in V(G)$ is called a *major* (*minor*) *vertex* if $\deg_G(x) \geqslant 3k+2$ ($\deg_G(x) \leqslant 3k+1$).

Note that if $M = M(G)$ denotes the plane graph induced by the set of major vertices of G, Properties 1–5 used to prove the upper bound in Theorem 1 can be deduced here in the same way. Moreover, we have

Property 6. *Each* 3-*face of* $M(G)$ *is a* 3-*face of* G (*because of the* 4-*connectivity of* G).

Clearly, $\delta_0 := \delta(M)$ is at most 5. Suppose, $0 \leqslant \delta_0 \leqslant 3$. Let $x \in V(M)$ be a vertex with $\deg_M(x) = \delta_0$. Applying Properties 4 and 5 we obtain $\deg_G(x) \leqslant \delta_0 + \delta_0(k-1) = \delta_0 k \leqslant 3k < 3k+2$, a contradiction.

Suppose, now, $4 \leqslant \delta_0 \leqslant 5$. From Lebesgue [12], we know that M either contains a vertex x with $\deg_M(x) = 4$ such that x is incident with at least one 3-face in M, or contains a vertex y with $\deg_M(y) = 5$ such that y is incident with at least four 3-faces in M.

Using Properties 4–6 we find $\deg_G(x) \leqslant 4 + 3(k-1) < 3k+2$ or $\deg_G(y) \leqslant 5 + k - 1 < 3k + 2$. Either case yields a contradiction. $\square$

Proof of Theorem 3. Consider any connected planar graph $H \neq P_k$ $(k \geqslant 1)$ and any integer $m \geqslant 4$. Let T be a triangulation of the plane where H is a subgraph of T with $V(T) = V(H)$. Insert into every triangle $\alpha = xyz \in F(T)$ three disjoint paths $P = [x = x_1, x_2, \ldots, x_m]$, $Q = [y = y_1, y_2, \ldots, y_m]$, $R = [z = z_1, z_2, \ldots, z_m]$ and, in addition, edges $x_1 z_i$, $y_1 x_i$, $z_1 y_i$ and $x_m y_i$, $y_m z_i$, $z_m x_i$ for $i = 2, \ldots, m$. The resulting graph G belongs to $\mathscr{G}(3,4;H)$. Every subgraph of G isomorphic to H has a vertex x such that $\deg_G(x) \geqslant m$ which proves the statement. $\square$

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